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Geometric Approach to Plane Trigonometry Using Classical Constructions

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ABSTRACT :

This document offers a geometric perspective on plane trigonometry, based on classical constructions using squares, circles, and triangles. Using only fundamental mathematical concepts, we are able to determine the exact values of the sine and cosine trigonometric functions for specific angles such as 15°, 30°, 45°, and 75° without a calculator. These results are obtained using construction techniques inspired in particular by the work of Abu al-Wafa. We then establish the laws of cosine using elementary geometric constructions and prove Mollweide's formulas based on the characteristics of isosceles triangles and the laws of sine. This approach highlights the link between Euclidean geometry and trigonometry, while proving to be a valuable educational tool for teaching at secondary and university level.

Keyword : Trigonometry, Geometric constructions, Trigonometric functions, Euclidean geometry.

1. INTRODUCTION

The origins of trigonometry (from the Greek trigonos, triangle) can be traced back to ancient Egypt, Mesopotamia, and the Indus Valley more than 4,000 years ago. The first use of sine was recorded in India between 800 and 500 BC [1]. It therefore plays an essential role in mathematics, both for its practical applications and for its importance in the analysis of geometric relationships. From a historical perspective, trigonometric functions emerged in response to geometric problems associated with triangles, circles, and angular measurements [2]. Euclid's seminal work laid the foundations of plane geometry, on which much of classical trigonometry is based [3].

In modern teaching, the values of the sine and cosine of notable angles are often presented as results to be memorized, without always highlighting their geometric origin. However, an approach based on construction not only provides a better understanding of these results, but also strengthens learners' logical reasoning and spatial visualization [4]. The geometric methods developed by mathematicians such as Abu al-Wafa perfectly illustrate this constructive approach [5].

The aim of this work is to show how standard trigonometric values can be determined from simple constructions involving squares, circles, and specific triangles. We also present a geometric demonstration of the cosine laws and Mollweide's formulas, in order to illustrate the internal consistency of plane trigonometry and its roots in Euclidean geometry [6,7].

2. Mathematical tools

The results used in this section are based primarily on classical Euclidean geometry as set out in Euclid's Elements [3] and reproduced in numerous modern geometry textbooks [4].

Proposition 1.1 :[3]

Let (C) be a circle with center$O$. Consider an arc $\overset{\frown}{AB}$ of this circle.

If M is a point on the circle (C) on the major arc $\overset{\smile}{AB}$ then $\widehat{AOB} = 2\widehat{AMB}$.

Sine theorem:[8]

Ic ABC is any triangle such that $AB = c$, $AC = b$, $BC = c$ and $\widehat{CBA} = \beta$, $\widehat{ACB} = \gamma$, $\widehat{BAC} = \alpha$. Then,

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

Theorem 1.1 (Pythagorean theorem) **:**

In a right triangle, the square of the length of the hypotenuse (or side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

This theorem remains a fundamental pillar of plane geometry. It is demonstrated in several documents such as [3,8] and remains central to the study of triangles and trigonometric constructions.

The two figures presented in this paper were constructed using Mathgraph32 dynamic geometry software [8]. This software was used to create precise Euclidean constructions involving specific squares, circles, and triangles.

The purpose of these constructions is to determine exact trigonometric values without using a calculator, and to demonstrate the laws of cosine and Mollweide's formulas, relying solely on geometric relationships between lengths and angles.

MathGraph32 is used here exclusively as a construction and visualization tool, allowing the geometric configurations necessary for the demonstrations to be clearly represented. The results obtained are based entirely on geometric and analytical arguments developed in the text, independently of the software.

3. Calculations of the sine and cosine functions of angles 15° ; 30° ; 45° ; 60° and 75° through a square and triangles

Using constructions based on squares, equilateral triangles, and isosceles right triangles, we geometrically determine the exact values of sine and cosine in angles of 15°, 30°, 45°, 60°, and 75°.

3.1 Method and construction

Method of construction 2.1 :

1. Construct a square $ABCD$ such that its diagonals intersect at point O and its sides measure a.

2. Draw a circle with center$A$ passing through K and two lines (D) and (D') tangent to the circle passing through point C .

3. Label E and F he two points of intersection of these lines and the square belonging to segments $[AD]$ and $[AB]$ respectively as. $EB = FD = x$

4. Connect points E, F and C. Line (EF) intersect (AC) at point H.

Construction 2.2 :

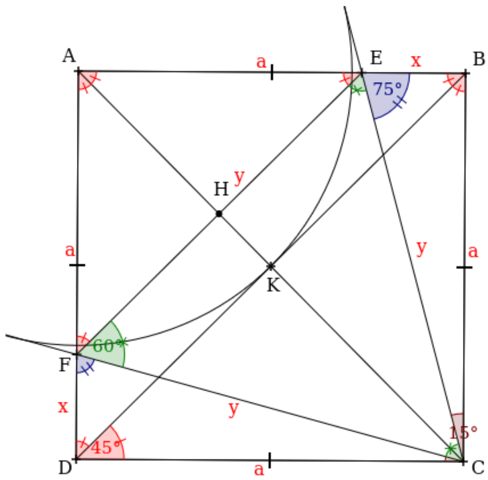

Figure 1 : Geometric construction enabling the determination of exact trigonometric values in specified angles

For a modern pedagogical approach to geometry, we used Mathgraph32 software [9] to graphically illustrate the theoretical results.

Thus, the calculations of the sine and cosine of the angles mentioned above derive directly from the properties of equilateral and isosceles right triangles, combined with the Pythagorean theorem.

3.2 Sine and cosine of 30° and 60°

According to Abūl Wafā's method, triangle EFC is equilateral i.e $EF = FC = CE = y$ and $\widehat{EFC} = \widehat{FCE} = \widehat{CEF} = 60°$. Since E andt F symmetrical with respect to (AC) and $(EF) \cap (AC) = H$, the intersection point H is the midpoint of $[EF]$. It also represents the foot of the height (and/or the bisector according to the properties of the equilateral triangle) from the vertex C of triangle EFC, so triangle EHC is a right triangle at H and according to the Pythagorean theorem, we have $\boxed{EC^2 = EH^2 + HC^2}$.

H is the midpoint of $[EF]$ implies that $HE = \frac{EF}{2} = \frac{y}{2}$ and (HC) the bisector originating from vertex C , of equilateral triangle EFC, cause $\widehat{HCE} = \frac{\widehat{FCE}}{2} = 30°$.

$$\text{As a result, } \sin(\widehat{HCE}) = \sin(30°) = \frac{HE}{EC} = \frac{\frac{y}{2}}{y} = \frac{1}{2}\text{; and}$$

$$\cos(\widehat{HCE}) = \cos(30°) = \frac{HC}{EC} = \frac{\sqrt{EC^2-EK^2}}{EC} = \frac{\sqrt{y^2-\left(\frac{y}{2}\right)^2}}{y} = \frac{\frac{y\sqrt{3}}{2}}{y} = \frac{\sqrt{3}}{2}.$$

$$\text{Sin}(\widehat{CEF}) = \sin(60°) = \frac{HC}{EC} = \frac{\frac{y\sqrt{3}}{2}}{y} = \frac{\sqrt{3}}{2}\text{ ; and } \cos(\widehat{CEF}) = \cos(60°) = \frac{HE}{EC} = \frac{1}{2}$$

3.3 Sine et cosine of angle 45°

Triangle BDC is an isosceles right triangle a C beacause $ABCD$ is a square, so $BC = DC = a$ and $\widehat{DCB} = 90°$. According to the properties of isosceles triangles, the angles at the base are equal.

Thus, $(\widehat{BDC} + \widehat{DCB} + \widehat{CBD} = 180°$ and $\widehat{BDC} = \widehat{CBD}) \Rightarrow \widehat{BDC} = \widehat{CBD} = 45°$.

$$\text{Cos}(\widehat{CBD}) = \cos(45°) = \frac{BC}{BD} = \frac{BC}{\sqrt{DC^2 + BC^2}} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin(\widehat{CBD}) = \sin(45°) = \frac{DC}{BD} = \frac{DC}{\sqrt{DC^2 + BC^2}} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

3.4 Sine and cosine of angles 15° and 75°

Points E and F are symmetrical with respect to the line (AC) and $ABCD$ is a square, so triangle EAF is an isosceles right triangle at A and $\widehat{FEA} = \widehat{AFC} = 45°$.

Since $\widehat{FEA} + \widehat{CEF} + \widehat{BEC} = 180°$ (because the three angles form a flat angle) and $\widehat{CEF} = 60°$, then $\widehat{BEC} = 180° - 45° - 60° = 75°$.

The ECB triangle is a right triangle at B (since $ABCD$ is a square) and $\widehat{ECB} + \widehat{CBE} + \widehat{BEC} = 180°$.

Thus, $\widehat{ECB} = 180° - (\widehat{CBE} + \widehat{BEC}) = 180° - (90° + 75°) = 15°$.

Furthermore, considering triangles EAF and BAD which are isosceles right triangles at A, we have:

$\begin{cases} AF = AD - FD \\ AE = AB - EB \end{cases} \Rightarrow \begin{cases} AF = a - x \\ AE = a - x \end{cases}$ and according to the Pythagorean theorem,

$$EF^2 = AF^2 + AE^2 \Longrightarrow y^2 = (a-x)^2 + (a-x)^2 \quad \Longrightarrow \; y^2 = 2(a-x)^2 \qquad (*)$$

In the ECB triangle, we have : $EC^2 = EB^2 + BC^2 \Longrightarrow y^2 = x^2 + a^2$ $(**)$.

By comparing $(*)$ and $(**)$, $x^2 + a^2 = 2(a-x)^2 \Longrightarrow x^2 - 4ax + a^2 = 0$

Let's determine x in terms of a (with $x < a$) using the discriminant : $\Delta = 12a^2 \Longrightarrow \sqrt{\Delta} = 2a\sqrt{3}$

$x = \frac{-4a-2a\sqrt{3}}{2} < 0$ or $= \frac{4a-2a\sqrt{3}}{2} > 0$, therefore $x = (2-\sqrt{3})a \Longrightarrow EB = (2-\sqrt{3})a$.

Replacing x with its expression in $(**)$ we obtain $EC^2 = \left((2-\sqrt{3})a\right)^2 + a^2 = (8-4\sqrt{3})a^2$

Therefore $EC = 2a\sqrt{2-\sqrt{3}}$.

As a result, $\sin\widehat{ECB} = \sin(15°) = \frac{EB}{EC} = \frac{(2-\sqrt{3})a}{2a\sqrt{2-\sqrt{3}}} = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ because $(\sqrt{6}-\sqrt{2})^2 = 4(2-\sqrt{3})$

$$\cos\widehat{ECB} = \cos 15° = \frac{BC}{EC} = \frac{a}{2a\sqrt{2-\sqrt{3}}} = \frac{\sqrt{2-\sqrt{3}}}{2(2-\sqrt{3})} = \frac{\sqrt{2-\sqrt{3}}(2+\sqrt{3})}{2} = \frac{(\sqrt{6}-\sqrt{2})(2+\sqrt{3})}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Finally, $\sin(75°) = \cos(15°) = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\cos(75°) = \sin(15°) = \frac{\sqrt{6}-\sqrt{2}}{4}$ because angles $\widehat{ECB}$ and $\widehat{BEC}$ are complementary.

4. The laws of cosine and Mollweide's formulas

4.1 Method and construction

Construction method 3.1 :

- Construct a triangle ABC such that $AB = c$, $AC = b$, $BC = c$ and $\widehat{CBA} = \beta$, $\widehat{ACB} = \gamma$, $\widehat{BAC} = \alpha$.
- Place a point D on the segment $[BC]$ such that $AC = b = CD$.
- Connect points A, D and B.
- Place E a point one the line (AC) located outside $[AC]$, on the side of point C, such that $CE = CB = a$.
- Connect point A, E and B.
- Draw the height from vertex B of triangle ABC.
- Label the foot of this height H.

Construction 3.2 :

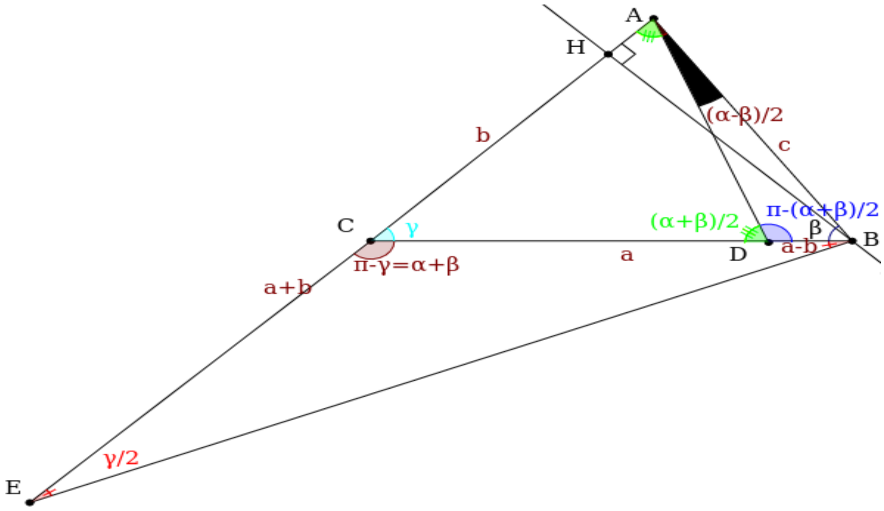

Figure 2 : Geometric construction used to demonstrate the cosine laws and Mollweide formulas

This figure allows us to clearly visualize the metric relationships between the sides and angles of the triangle, facilitating understanding of the cosine law, Mollweide's formula, and their geometric proofs.

4.2 The laws of cosine

Based on the height diagram in any triangle and the application of the Pythagorean theorem to the right-angled triangles obtained, we demonstrate the law of cosines.

In triangle ABC, point H is the foot of the height from vertex B, so lines (BH) and (AC) are perpendicular (by definition of a height in a triangle).

Thus, triangles ABH and HBC are right-angled at H, so $AH = AC - HC$ and, according to Pythagoras' theorem, $AB^2 = BH^2 + AH^2$.

As a result, $\sin\gamma = \frac{BH}{BC} \Longrightarrow BH = BC\sin\gamma$ and $\cos\gamma = \frac{HC}{BC} \Longrightarrow HC = BC\cos\gamma$;

$$AB^2 = BH^2 + AH^2 \Longrightarrow AB^2 = (BC\sin\gamma)^2 + (AC - BC\cos\gamma)^2$$

$$\Longrightarrow c^2 = (a\sin\gamma)^2 + (b - a\cos\gamma)^2$$

$$\Longrightarrow c^2 = a^2 + b^2 - 2ab\cos\gamma$$

To find the other two formulas, simply follow the same procedures by drawing the heights from vertices A and C .

4.3 Mollweide's formula

We derive these formulas from the properties of isosceles triangles and the laws of sines. They are commonly used as a verification tool in trigonometry and appear in many classical and modern treatises [5,6].

Triangle ACD is isosceles at C because $CD = AC = b$, therefore $\widehat{DAC} = \widehat{CDA} = \frac{\alpha+\beta}{2}$.

In triangle ADB, we have : $\widehat{BAD} = \frac{\alpha-\beta}{2}$, $\widehat{DBA} = \beta$ and $\widehat{ADB} + \widehat{DBA} + \widehat{BAD} = 180°$ therefore $\widehat{ADB} = 180° - \left(\widehat{DBA} + \widehat{BAD}\right) = 180° - \frac{\alpha+\beta}{2}$.

Since $CE = BC = a$, triangle CEB is isosceles at C and the angles at the base are equal, i.e., $\widehat{CEB} = \widehat{EBC}$. Since angle $\widehat{ACE} = \widehat{ACB} + \widehat{BCE} \Rightarrow \widehat{BCE} = \widehat{ACE} - \widehat{ACB} = 180° - \gamma$ et $\widehat{CEB} + \widehat{EBC} + \widehat{BCE} = 180° \Rightarrow \widehat{CEB} = \widehat{EBC} = \frac{\gamma}{2}$.

Therefore, according to the laws of sine, in triangle ADB, we have :

$$\frac{BD}{\sin \widehat{BAD}} = \frac{AB}{\sin(\widehat{ADB})} \Rightarrow \frac{a-b}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{c}{\sin\left(180°-\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \frac{a-b}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{c}{\sin\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \frac{a-b}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{c}{\sin\left(90°-\frac{\gamma}{2}\right)}$$

$$\Rightarrow \frac{a-b}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{c}{\cos\left(\frac{\gamma}{2}\right)}$$

$$\Rightarrow (a-b)\cos\left(\frac{\gamma}{2}\right) = c\sin\left(\frac{\alpha-\beta}{2}\right)$$

In triangle AEB, $\widehat{CEB} = \widehat{AEB}$ and angle $\widehat{EBA} = \widehat{EBC} + \widehat{CBA} = \beta + \frac{\gamma}{2} \Rightarrow \widehat{EBA} = \beta + 90° - \frac{\alpha+\beta}{2} = 90° - \frac{\alpha-\beta}{2}$

Applying the law of sines, we have : $\frac{AE}{\sin \widehat{EBA}} = \frac{AB}{\sin \widehat{AEB}} \Rightarrow \frac{a+b}{\sin\left(90°-\frac{\alpha-\beta}{2}\right)} = \frac{c}{\sin\left(\frac{\gamma}{2}\right)}$

$$\Rightarrow \frac{a+b}{\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{c}{\sin\left(\frac{\gamma}{2}\right)}$$

$$\Rightarrow c\cos\left(\frac{\alpha-\beta}{2}\right) = (a+b)\sin\left(\frac{\gamma}{2}\right)$$

5. Comparative discussion

The exact trigonometric values and fundamental identities of plane trigonometry are well established in classical literature. They are generally obtained by algebraic or analytical methods, and often presented without detailed geometric proof.

This work takes a different approach, favoring explicit geometric constructions that allow exact angle values to be calculated without the use of numerical tools. Furthermore, using Euclidean figures, we have demonstrated the laws of cosines and Mollweide's formulas.

Almost similar geometric approaches appear in some classical works, but they are rarely developed in a unified and systematic manner. The interest of this work thus lies in the direct relationship between simple geometric constructions and fundamental trigonometric results, highlighting the internal consistency of plane trigonometry.

This work is limited to Euclidean plane trigonometry and specific geometric configurations. The proposed constructions concern only common trigonometric values and do not cover all possible angles.

Furthermore, no extension to more general frameworks, such as spherical trigonometry or advanced analytical methods, is envisaged.

Finally, although figures allow us to obtain exact values and rigorous proofs, this approach remains essentially geometric and is not intended to replace traditional analytical methods, but rather to complement them.

6. Conclusion

In this article, we have highlighted the effectiveness of geometric constructions for studying trigonometric functions and the fundamental relationships between the sides and angles of a triangle. Determining the exact values of the sine and cosine of the angles mentioned, without a calculator, using the square and the equilateral triangle provides a better understanding of the origin of these results, which are often perceived as purely algebraic.

Geometric proofs of the cosine law and Mollweide's formulas show that these relationships are not simply formulas to be applied, but natural consequences of the properties of triangles and elementary trigonometric laws. This approach is consistent with classical work in Euclidean geometry.

In conclusion, these results constitute a relevant teaching aid for trigonometry, as they promote geometric intuition, deductive reasoning, and a deep understanding of mathematical concepts at both the secondary and university levels.

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