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The Maximum Flow Problem of Uncertain Network with Respect to Uncertain Loss

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ABSTRACT

Uncertainty theory provides a new tool to deal with the in deterministic factors. The purpose of this paper is to study the maximum flow problem of the uncertain network with respect to uncertain loss by using the uncertainty theory. In this paper, the uncertainty distribution of the lost flow was given. With the help of the operational law of uncertainty theory, a mathematical model of the maximum flow problem of the uncertain network with respect to uncertain loss has been established and a solution was carried out.

1 INTRODUCTION

The maximum flow problem is one of the classic combinatorial optimization problems. The problem is to find a flow of maximum value on a network from a source to a sink. The maximum flow problem originated from the Soviet railway system [1]. Ford and Fulkerson [2] mentioned that the maximum flow problem was formulated by T.E. Harris as follows, Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.

The maximum flow problem is one of the classical problems of network optimization and it has been widely used in every field such as electrical powers, traffics, communications, computer networks and logistics. Therefore, it is important to study this problem both from theoretical and practical points of view. Nowadays, there have emerged many efficient algorithms for the maximum flow problem for over five decades [3]-[9]. For a fixed network, if the arc capacities of the network are given, then we can calculate the maximum flow. In the classical maximum flow problem, we require knowing the arc capacities of the network, demand flow conservation of each node besides the source, and sink. This means that material cannot accumulate, be created, dissipate, or be lost at any node besides the source or the sink. However, due to some reasons, different types of uncertainty are frequently encountered in practice and must be taken into account. Especially, for a network in the real world, not only we do not know the capacity of each arc, but also the flow is not conserved for some node in the network. This means that the flow can be lost at some node in the network. In addition, sometimes we do not know how much flow has been lost, which causes that the traditional methods cannot be used to verify some of the properties of the maximum flow problem. Some researchers believe that these belong to the randomness factor, and they used the probability theory to study these uncertain phenomena, such as the Nawathe and Rao [10]. At the same time, some researchers adopt the fuzzy theory to deal with this matter and have made many studies [11]-[14].

However, the premise of application of probability theory is that we have obtained the probability distribution of the must fully close to the real frequency. Unfortunately, we often face problems about the lack of observation data, so we cannot calculate the frequency of the event and cannot determine a probability distribution. In this case, we have to estimate the possible reliability of event based on experience and knowledge of experts. People often overestimate an unlikely event, which makes the reliability far greater than the variance of frequency. At this time, if the reliability is considered subjective probability, the deduced results will be varied widely compared with what we have expected. In order to research the subjective indeterminacy phenomenon, uncertainty theory was founded by Liu [15] in 2007 and refined by Liu [16] in 2010. It becomes an axiomatic branch of mathematics and has been put into a series of successful applications, which provides a motivation to the uncertainty theory of the introduction of the maximum flow problem. Uncertainty theory provides a new approach to the study of the maximum flow problem. In this paper, we will study the maximum flow problem with an uncertain loss by uncertainty theory. In addition, we can see that uncertainty theory offers a powerful tool to deal with this problem.

2 PRELIMINARIES

Uncertainty theory, founded by Liu [15] in 2007 and refined by Liu [16] in 2010, provides a new approach to deal with indeterminacy factors. Nowadays it has become a branch of mathematics based on normality, duality, subadditivity, and product axioms. So far, the theory and practice have shown that the uncertainty theory is a very effective tool to deal with indeterminacy information, in particular, the empirical data and subjective estimates.

Here we briefly introduce the major developments of uncertainty theory in different areas. Liu [17] introduced the uncertain process and offered the definition of differential equations. In 2010, Liu [18] offered the uncertain set theory and uncertainty reasoning method containing new inference rules. In 2009, Liu [19] proposed uncertain programming, that is, the mathematical programming with uncertain variables. Gao [20] in 2009 proved some properties of the continuous uncertain measure. Gao et al. [21] discussed Liu's inference rule with multiple antecedents and with multiple if-then rules in 2010. In 2011, the concepts of the uncertainty graph and connectivity index of uncertainty graph in [22]. In 2012 Gao [23, 24] gave offered the concepts of Cycle index of the uncertainty graph and Tree index of uncertainty graph. You [25] provided some uncertainty convergence theorem. Gao [26] studied Shortest Path Problem with Uncertain Arc Lengths in 2011. Based on uncertainty theory, some significant and theoretical work of uncertainty theory such as uncertain calculus [27], uncertain differential equation [28], uncertain logic [29], and uncertain risk analysis [30] has been established. In short, the researchers and applications of uncertainty theory have been increasingly extended. The readers may consult [31] to learn the recent developments of the uncertainty theory.

Now, we introduce the concepts and results of this uncertainty theory applied in this paper.

Let Γ be a nonempty set, and L a σ-algebra over Γ. Each element $\Lambda \in L$ is called an event. The set function M is called an uncertain measure if it satisfies the following three axioms [15]:

Axiom 1. (Normality) $\mathrm{M}\{\Gamma\}=1.$

Axiom 2. (Duality) $\mathrm{M}\{\Lambda\}+\mathrm{M}\{\Lambda^c\}=1$ for any event Λ.

Axiom 3. (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}M\{\Lambda_i\}$$

The triplet (Γ, L, M)s called an uncertainty space. An uncertain variable is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of real numbers.

Product uncertain measure was defined by Liu [27] in 2009, producing the fourth axiom as follows

Axiom 4. (Product Axiom) Let (Γ_k, L_k, M_k) be uncertainty spaces for $k=1,2,\cdots$. then the prod- uct uncertain measure M is an uncertain measure on the product σ-algebra

$L = L_1 \times L_2 \times \cdots$ satisfying

$$M\{\prod_{k=1}^{\infty} \Lambda_k\} \leq \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events for $L_k, k = 1, 2, \cdots,$ respectively?

The uncertain variables $\xi_1, \xi_2, \cdots \xi_m$ are said to be independent if

$$M\{\bigcap_{i=1}^{m} \xi_i \in B_i\} \leq \bigwedge_{i=1}^{m} M\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Definition 1 (Liu [15]) The uncertainty distribution $\Phi : R \to [0,1]$ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathrm{M}\{\xi \leq X\}.$$

Remark 1 The uncertainty distribution $\Phi(x)$ is an increasing function.

Definition 2 (Liu [15]) An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$.

In this paper, we assume all given uncertainty distributions are regular. Otherwise, we may give the uncertainty distribution a small perturbation such that it becomes regular. To calculate the uncertain measure from an uncertainty distribution, Liu [16] presented the measure inversion theorem, that is

$$\mathrm{M}\{\xi \leq X\} = \Phi(x), \qquad \mathrm{M}\{\xi \geq X\} = 1 - \Phi(x).$$

A real-valued function $f(x_1, x_2, \cdots x_n)$ is said to be strictly increasing if f satisfies the following conditions

(1) If $x_i \le y_i$ for $i = 1, 2, \cdots, n$, then $f(x_1, x_2, \cdots x_n) \le f(y_1, y_2, \cdots y_n)$;

(2) If $x_i < y_i$ for $i = 1, 2, \cdots, n$, then $f(x_1, x_2, \cdots x_n) < f(y_1, y_2, \cdots y_n)$;

Definition 3 (Liu [15])Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{M}\{\xi \ge r\}dr - \int_{-\infty}^{0} \mathrm{M}\{\xi \le r\}dr$$

provided that at least one of the two integrals is finite.

In order to calculate the expected value via inverse uncertainty distribution, Liu [16] proved that

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

Meanwhile, Liu [16] proved the linearity of expected value operator, that is

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

for two independent uncertain variables ξ, η and two crisp numbers a, b.

3 THE MAXIMUM FLOW MODEL WITH THE UNCERTAIN LOSS

In general, a deterministic network is denoted as $N = (V, A)$, where $V = \{1, 2, \cdots n\}$. is a finite set of nodes, and $A = \{(i, j) | i, j \in V\}$ is the set of arcs. In this paper, we suppose that the networks are single-source, single-sink with n nodes and m arcs. A a network of 6 nodes and 8 arcs is shown in Figure 1, where node 1 is source and node 6 is the sink.

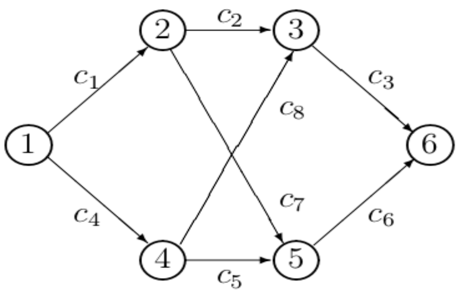

Figure 1

Let c_{ij} denote the arc capacities from node i to node j, and x_{ij} denote the flow from node i to node j, u_i denote the flow of being lost at the node i, $i, j = 1, 2, \cdots n$, s denote the source node and t is the sink node. Then the maximum flow problem with the loss can be formulated as follows:

$$\begin{cases} \max \quad v \\ \text{subject to:} \\ \sum_j (x_{sj} - x_{js}) = v, \\ \sum_j (x_{ij} - x_{ji}) = -v + \sum_{i \neq s,t} u_i, \\ \sum_j (x_{ij} - x_{ji}) \geq -u_i, \ i \in N, \ i \neq s,t, \\ 0 \leq x_{ij} \leq c_{ij}, \qquad (i,j) \in A. \end{cases} \tag{1}$$

In the above model, the quantities c_{ij} u_i, are all assumed to be crisp numbers. However, sometimes the network plan is made in advance, so the quantities generally are not fixed but obtained from experience evaluation or expert knowledge. In this case, we may assume the quantities are uncertain variables. Then the model (1) is only a conceptual model rather than a mathematical model because there does not exist a natural order relation in an uncertain world. Here we take expected value criterion or confidence level of the constraint functions. Then the model (1) turns into the following mathematical model:

$$\begin{cases} \max \quad v \\ \text{subject to:} \\ \sum_j (x_{sj} - x_{js}) = v, \\ \sum_j (x_{tj} - x_{jt}) = -v + E\left[\sum_{i \neq s,t} u_i\right], \\ \mathrm{M}\left\{\sum_j (x_{ij} - x_{ji}) \geq -u_i\right\} \geq \beta_i \qquad i \in N, i \neq s,t, \\ \mathrm{M}\left\{x_{ij} \leq c_{ij}\right\} \geq \gamma_{ij}, \qquad (i,j) \in A \\ x_{ij} \geq 0 \end{cases} \tag{2}$$

Where β_{ij}, γ_{ij} are some predetermined confidence levels $i \neq s,t,(i,j) \in A$. The following theorem shows that the model (2) is equivalent to a determined model, for which many efficient algorithms have been designed.

Theorem 1. Let that c_{ij}, u_i be independent uncertain variables with uncertainty distributions Φ_{ij}, Ψ_i. Then the model (2) is equivalent to the following model:

$$\begin{cases} \max \quad v \\ \text{subject to:} \\ \sum_j (x_{sj} - x_{js}) = v, \\ \sum_j (x_{tj} - x_{jt}) = -v + \sum_{i \neq s,t} \int_0^1 \Phi_i^{-1}(\alpha) d\alpha, \\ \sum_j (x_{ij} - x_{ji}) \geq \Psi_i^{-1}(\beta_i), \qquad i \in N, i \neq s,t, \\ x_{ij} \leq \Phi_{ij}^{-1}(1-\gamma_{ij}), \qquad (i,j) \in A \\ x_{ij} \geq 0 \end{cases} \tag{3}$$

Proof It follows from the linearity of expected value operator that

$$E[\sum_{i \neq s,t} u_i] = \sum_{i \neq s,t} E[u_i]$$

for the independent uncertain variables $u_i, i \in N, i \neq s,t.$

Since,

$$E[u_i] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

then, we have

$$E[\sum_{i \neq s,t} u_i] = \sum_{i \neq s,t} \int_0^1 \Phi_i^{-1}(\alpha)d\alpha.$$

Since u_i and c_{ij} have uncertainty distributions Ψ_i and Φ_{ij}, due to the measure inversion theorem, we have

$$\mathrm{M}\left\{\sum_j (x_{ij} - x_{ji}) \geq -u_i\right\} \geq \beta_i \quad i \in N, i \neq s,t,$$

which is equivalent to

$$\sum_j (x_{ij} - x_{ji}) \geq \Psi_i^{-1}(\beta_i), \quad i \in N, i \neq s,t,$$

the following inequality

$$\mathrm{M}\left\{x_{ij} \leq c_{ij}\right\} \geq \gamma_{ij}, \qquad (i,j) \in A$$

is equivalent to

$$x_{ij} \leq \Phi_{ij}^{-1}(1-\gamma_{ij}), \qquad (i,j) \in A$$

Then, the poor is completed.

4 NUMERICAL EXPERIMENT

In this section, we will give an example of how to calculate the maximum flow of uncertain net-

work with the uncertain loss.

Example 1 Let G be a directed network with five arcs defined by Figure 2.

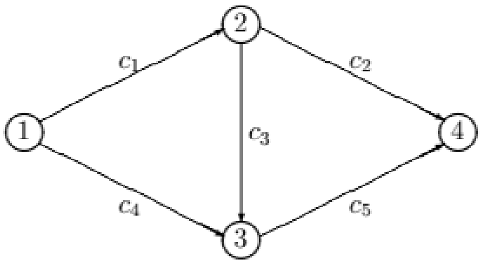

Figure 2: Network G for Example 1

In an uncertain environment, we assume that the flow of being lost on the i-the node are uncertain variables ξ_i with linear uncertainty distributions $\Phi_i, i = 2,3$;,respectively.

$$\Phi_2(x) = \begin{cases} 0, & if \quad x < 2, \\ (x-2)/2, & if \ 2 \le x \le 4 \\ 1, & if \quad x > 4. \end{cases}$$

$$\Phi_3(x) = \begin{cases} 0, & if \quad x < 1, \\ (x-1)/2, & if \ 1 \le x \le 3 \\ 1, & if \quad x > 3. \end{cases}$$

And we assume that the capacities of the (i, j)-arc are uncertain variables η_{ij} with linear uncertainty distributions $\Psi_{ij}, i \ne j; i, j, = 1,2,3,4,5,$ respectively.

$$\Psi_{12}(x) = \begin{cases} 0, & if \quad x < 8, \\ (x-8)/4, & if \ 8 \le x \le 12 \\ 1, & if \quad x > 12. \end{cases}$$

$$\Psi_{24}(x) = \begin{cases} 0, & if \quad x < 5, \\ (x-5)/4, & if \ 5 \le x \le 9 \\ 1, & if \quad x > 9. \end{cases}$$

$$\Psi_{23}(x)=\begin{cases} 0, & if \quad x<1, \\ (x-1)/4, & if\ 1\le x\le 5 \\ 1, & if \quad x>5. \end{cases}$$

$$\Psi_{13}(x)=\begin{cases} 0, & if \quad x<4, \\ (x-4)/4, & if\ 4\le x\le 8 \\ 1, & if \quad x>8. \end{cases}$$

$$\Psi_{34}(x)=\begin{cases} 0, & if \quad x<7, \\ (x-7)/4, & if\ 7\le x\le 11 \\ 1, & if \quad x>11. \end{cases}$$

It follows from Theorem 1 that the model (3) is equivalent to the following model,

$$\begin{cases} \max \quad v \\ \text{subject to}: \\ \sum_j (x_{1j} - x_{j1}) = v, \\ \sum_j (x_{4j} - x_{j4}) = -v + E\left[\sum_{i\neq 1,4} u_i\right], \\ \mathrm{M}\left\{\sum_j (x_{ij} - x_{ji}) \ge -u_i\right\} \ge \beta_i \qquad i\in N, i=2,3, \\ \mathrm{M}\left\{x_{ij}\le c_{ij}\right\} \ge \gamma_{ij}, \qquad (i,j)\in A \\ x_{ij}\ge 0 \end{cases} \tag{4}$$

Note that the linear uncertain variable $L(a,b)$ has an expected value $\frac{a+b}{2}$ and an inverse uncertainty distribution

$$\Phi^{-1}(\alpha)=(1-\alpha)a+\alpha b.$$

Then, above model could be transformed into the following model,

$$\begin{cases} \max \quad v \\ \text{subject to:} \\ \sum_j (x_{1j} - x_{j1}) = v, \\ \sum_j (x_{4j} - x_{j4}) = -v + 5, \\ \sum_j (x_{ij} - x_{ji}) \geq (1-\beta_i)a_i + \beta_i b_i \quad i = 2,3, \\ x_{ij} \leq \gamma_{ij} a_{ij} + (1-\gamma_{ij}) b_{ij}, \qquad (i,j) \in A \\ x_{ij} \geq 0 \end{cases} \tag{5}$$

This model is a typical linear programming problem and its feasible region is a convex polyhedron. Assume the confidence levels are $\beta_i = 0.5$, $\gamma_{ij} = 0.75$. By means of the simplex method, the maximum flow for the network in Figure 2 is 14.

5 CONCLUSION

This paper mainly investigated a new maximum flow model based on the uncertainty theory. It was transformed into a deterministic model by taking expected value or confidence level of the constraint functions. A numerical example was also given and the optimal solution was obtained by the simplex method.

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