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Construction of Variance and Efficiency Balanced Designs using 3^n-Factorial Design

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ABSTRACT

A method of constructing equireplicate Variance Balanced (VB) and Efficiency Balanced (EB) design with unequal block sizes is proposed using 3^n-symmetrical factorial design with illustration. The method suggested here is based on merging some of the treatment combinations and deleting control and some other unimportant treatment combinations in 3^n-factorial design. Further optimality of the constructed design has been checked and found it to be universally optimal.

1 INTRODUCTION

We always need to setup a design in such a way that the variability in response due to uncontrolled variables (sometimes called experimental error) is less. We also want designs which are efficient, i.e., designs where we can answer the questions of interest with a minimal amount of data because of the expense associated with data collection. Especially in agricultural experiments, we preferred incomplete block designs when experimental units are very large. In many experimental situations, it is a severe restriction that all blocks in the experiment are of the same size. Variance Balanced (VB) designs forms a class of designs that are flexible extensions to balanced incomplete block designs. They provide the ability to design an experiment with equal precision among all pairwise comparisons, without being restricted to equal block size and equal replication of the treatments.

The importance of VB designs in the context of experimental material is well known as it yields optimal design apart from ensuring simplicity in the analysis. Many practical situations demand designs with varying block sizes (Pearce [1964]), or resolvable VB designs with unequal replications (Kageyama [1976], Mukerjee and Kageyama [1985]).

Rao [1958] gave the necessary and sufficient condition for a block design to be variance-balanced. Pearce [1964] observed that it is sufficient to ensure the constancy of off-diagonal elements of the matrix C (= $R - NK^{-1}N'$) for a design to have variance balance. Hedayat and Federer [1974] defined that a design is said to be VB if every normalized estimable linear function of treatment effect can be estimated with the same precision. Designs require an equal number of replications on all the treatments and equal block sizes. These two conditions were relaxed with the introduction of new class of balanced designs called VB designs.

A block design is called variance-balanced if and only if

1. It permits the estimation of all normalized treatment contrasts with the same variance.

2. If the information matrix for treatment effects $C = R - NK^{-1}N'$ satisfies $C = \psi\,[I_v - (1/v)\,1_v 1_v']$.

Where ψ is the unique nonzero eigenvalue of the matrix C with the multiplicity $(v - 1)$, I_v is the $v \times v$ identity matrix.

Hedayat and Federer [1974], Khatri [1982], Agarwal and Kumar [1984, 1985] have provided several methods for construction of VB designs. Kageyama [1988] gave some methods for constructing block designs with unequal treatment replications and unequal block sizes. Das and Ghosh [1985] have defined generalized efficiency balanced (GEB) designs which include both, VB as well as efficiency balanced (EB) designs. Ghosh et al. [1992] gave methods to construct binary and non-binary VB design.

The concept of efficiency balanced (EB) was introduced by Jones [23] and the nomenclature "efficiency balance" is due to Puri and Nigam [1975], Williams [1975]. Calinski [1971], Puri and Nigam [1975] established a sufficient condition for a design to be efficiency balanced is that its M matrix.

A block design is called efficiency balanced if

1. Every contrast of treatment effects is estimated through the design with the same efficiency factor.

2. $M = \mu I_v + (1-\mu)\, J_v\, r' / n$; (See Caliski [1971])

Where μ is the unique non zero eigenvalue of M with multiplicity (1-ν). For the EB block design N, the information matrix C is given as C = (1- μ) (R – (1/n) r r'); (see Kageyama [1974]).

Mukerjee and Saha [1990] derived some optimality results on efficiency balanced designs. Gupta et al. [1983] gave a method for constructing general efficiency balanced designs with equal and unequal block sizes. Gupta [1992] gave a method for constructing efficiency balanced designs through BIB and GD designs. Ghosh et al. [1994] introduced efficiency balanced and variance-balanced ternary block designs. Ceranka and Graczyk [2009] discussed some problems for a class of EB block design based on balanced incomplete block designs with repeated blocks. Sun and Tang [2010] gave the efficiency balanced designs and their constructions.

The idea of merging the levels of factors was exploited by Addelman [1962] to construct orthogonal resolution-III plans for symmetrical and asymmetrical factorial experiments. The effect of merging of treatments in block designs was studied by Pearce [1971] and it was shown that merging of treatments in general leads to an increase in precision of the resulting design if the merging has any effect.

In a given class of designs, one should attempt to choose a design which is good according to some well-defined statistical criterion. This has led to the study of optimality of experimental designs. Optimal designs are experimental designs that are generated based on a particular optimality criterion and are generally optimal only for a specific statistical model. The optimality of a design depends on the statistical model and is assessed with respect to a statistical criterion, which is related to the variance-matrix of the estimator.

Kiefer [1958] introduced Balanced Block Designs (BBD) as a generalization of Balanced Incomplete Block (BIB) designs and proved the A-, D- and E-optimality of BBD's in D (*v*, *b*, *k*), where D (*v*, *b*, *k*) is the class of all connected block designs with *v* treatments, *b* blocks, and constant block size *k*.

Let C_P denote the class of all acceptable designs with reference to *P*. C_P consists of only connected designs. For any design $d \in C_P$; let V_d denote the dispersion matrix, using *d*. Then

A- optimality A design $d^* \in C_P$ is said to be A-optimal in C_P if

$$tr\,(V_{d^*}) \le tr\,(V_d)$$

i.e. A-optimality criterion seeks to minimize the trace of the inverse of the information matrix. This criterion results in minimizing the average variance of the estimates of the regression coefficients.

D- optimality A design $d^* \in C_P$ is said to be D-optimal in C_P if

$$\det\,(V_{d^*}) \le \det\,(V_d)$$

i.e. D-optimality criterion seeks to minimize $|(X'X)^{-1}|$, or equivalently maximize the determinant of the information matrix X'X of the design.

E- optimality A design $d^* \in C_P$ is said to be E-optimal in C_P if

$$\max\,(\lambda_{d^*}) \le \max\,(\lambda_d)$$

i.e. E-optimality criterion seeks to maximize the minimum eigenvalue of the information matrix.

In fact, Subsequently, Kiefer [1975] proved a stronger result regarding the optimality of balanced block designs, by introducing the concept of *universal optimality* of BBD's in D (*v*, *b*, *k*). He obtained a sufficient condition for universal optimality and proved that the balanced block design (if it exists) is universally optimal in the class of all connected designs. If a design is A-, D- and E-optimal then it is universally optimal as well and a vice a versa.

Although a considerable amount of work is available on optimality of designs in D (*v*, *b*, *k*), not much appears to have been done on the optimality of designs with unequal block sizes, except by Lee and Jacroux [1987a,b,c], Dey and Das [1989], Gupta and Singh [1989], Gupta et al. [1991].

Factorial experiments are experiments that investigate effects of two or more factors or input parameters on the output response of a process. It involves simultaneously more than one factor each at two or more levels. Several factors affect simultaneously the characteristic under study in factorial experiments and the experimenter is interested in the main effects and the interaction effects among different factors. Experiments, in which the numbers of levels of all the factors are same, are called symmetrical factorial experiments. A full factorial experiment is an experiment whose design consists of two or more factors, each with discrete possible values or "levels", and whose experimental units take on all possible combinations of these levels across all such factors.

If there are k factors, each at 3 levels, a full factorial design has 3^k-runs. The three-level design is written as a 3^k-factorial design. It means that k factors are considered, each at 3 levels. These are (usually) referred to as low, intermediate and high levels. These levels are numerically expressed as 0, 1, and 2. A third level for a continuous factor facilitates investigation of a quadratic relationship between the response and each of the factors. A design with all possible high/intermediate/low combinations of all the input factors is called a 3^k-symmetric full factorial design in three levels.

Rajarathinam et al. [2014, 2016] gave the construction of unequal block sizes and equi-replicated binary variance-balanced and efficiency balanced designs from symmetrical 2^n-factorial design. Since they delete the control treatment and merged all the main effects and considered them as one block in the first method (See Rajarathinam et al. [2014]) and in the second method delete the control treatment as well as all the main effects in 2^n-factorial design (See Rajarathinam et al. [2016]). Here in our research, we construct equi-replicated binary variance-balanced and efficiency balanced designs from symmetrical 3^n-factorial design with unequal block sizes. The constructed designs discussed here based on merging first highest order linear effect with each first level main effects separately and deleting control and some other unimportant treatment combinations in 3^n-factorial design which are of less important in the practical point of view.

2 METHOD OF CONSTRUCTION

Let as consider a 3^n-factorial design. There are 3^n-treatment combinations where n-first level main effects and n-second level main effects are there in the design. Now delete the control treatment as well as n-second level main effects and merge the first highest order linear effect with each first level main effects separately. Thus we get $[3^n - (n+1)]$ treatment combinations. Now consider these $[3^n - (n+1)]$ treatment combinations as blocks for the required design with unequal (varying) block sizes.

For example, let n = 3, Then in 3^3-factorial experiment there are $3^3 = 27$ treatment combinations in all, which are as follows

"1"	=	0	0	0
a	=	1	0	0
b	=	0	1	0
c	=	0	0	1
bc	=	0	1	1
ac	=	1	0	1
ab	=	1	1	0
abc	=	1	1	1
a^2	=	2	0	0
b^2	=	0	2	0
c^2	=	0	0	2
a^2b^2	=	2	2	0
a^2c^2	=	2	0	2
b^2c^2	=	0	2	2
$a^2b^2c^2$	=	2	2	2
a^2b	=	2	1	0
ab^2	=	1	2	0
a^2c	=	2	0	1
b^2c	=	0	2	1
ac^2	=	1	0	2
bc^2	=	0	1	2
a^2bc	=	2	1	1
ab^2c	=	1	2	1
abc^2	=	1	1	2
a^2b^2c	=	2	2	1
a^2bc^2	=	2	1	2
ab^2c^2	=	1	2	2

Delete a treatment combination whose levels of all factors are zero i.e. delete the control treatment. Also, delete the treatment combinations where the level of only one factor is two while the levels of the other factors are zero i.e. all the quadratic (second level) main effects. Next merge the first (intermediate) level highest order linear treatment combination with each first level main effect i.e. the treatment combination where the level of only one factor is one while the levels of the other factors are zero, separately. Here there are three first level linear main effects in 3^3-factorial experiment. The remaining treatment combinations remain as it is. Thus we get

1	1	0
1	0	1
0	1	1
1	1	1
2	1	1
1	2	1
1	1	2
2	2	0
2	0	2
0	2	2
2	2	2
2	1	0
1	2	0
2	0	1
0	2	1
1	0	2
0	1	2
2	1	1
1	2	1
1	1	2
2	2	1
2	1	2
1	2	2

Finally we get 3^3-(3+1) = 23 treatment combinations. Now transposing all the treatment combinations and treated them as blocks; we get the incidence matrix of variance-balanced and efficiency balanced design, with block sizes 2, 3, 4, 5 and 6 respectively.

Theorem 2.1 The existence of 3^n- symmetric factorial experiment implies the existence of equi-replicated variance-balanced and efficiency balanced design with unequal block sizes, having parameters

$v^* = n$, $b^* = 3^n - (n+1)$, $r^* = 3^n + (n-2)$, $k^* =$ [2,2,….,2; 3,3,….,3; ……; n, n,…., n; (n+1), (n+1),…., (n+1); (n+2), (n+2),…., (n+2); ……; (n+n), (n+n),…., (n+n)]

Proof In 3^n-symmetric factorial experiment, there are 3^n- treatment combinations in all. Considering "n" factors as rows and 3^n-treatment combinations as columns. Now delete the control treatment and the treatment combinations where the level of only one factor is two while the levels of the other factors are zero i.e. all the quadratic (second level) main effects. Next merge the first (intermediate) level highest order linear treatment combination with each first level main effect separately, we get the $[3^n-(n+1)]$ treatment combinations (which are treated as blocks); then incidence matrix N^* of design D^* with unequal block sizes is given as

$$N^* = \left[\begin{array}{ccccc:ccccc:c:ccccc:c:ccccc}
\multicolumn{5}{c}{\overbrace{\hphantom{1\ 1\ \cdots\ \cdots\ 0}}^{2-\text{factor interaction}}} & \multicolumn{5}{c}{\overbrace{\hphantom{1\ 1\ \cdots\ \cdots\ 0}}^{3-\text{factor interaction}}} & & \multicolumn{5}{c}{\overbrace{\hphantom{1\ 1\ \cdots\ \cdots\ 0}}^{(n-1)-\text{factor interaction}}} & \overbrace{}^{n-\text{factor interaction}} & \multicolumn{5}{c}{\overbrace{\hphantom{2\ 1\ \cdots\ \cdots\ 1}}^{{}^nC_1 \text{ blocks occurs due to merging of main effects with } n-\text{factor interaction}}} \\
1 & 1 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 0 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 0 & 1 & 2 & 1 & \cdots & \cdots & 1 \\
1 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 0 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 2 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 1 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 1 & \cdots\cdots & 1 & 0 & \cdots & \cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 1 & \cdots\cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & 2 \\
\multicolumn{22}{c}{\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}_{\text{Linear Effect}}}
\end{array}\right]$$

$$
\underbrace{
\begin{array}{ccccc|ccccc|c|ccccc|c}
\multicolumn{5}{c}{\overbrace{\hphantom{2\ 2\ \cdots\ \cdots\ 0}}^{2-factor\ interaction}} & \multicolumn{5}{c}{\overbrace{\hphantom{2\ 2\ \cdots\ \cdots\ 0}}^{3-factor\ interaction}} & & \multicolumn{5}{c}{\overbrace{\hphantom{2\ 2\ \cdots\ \cdots\ 0}}^{(n-1)-factor\ interaction}} & \overbrace{}^{n-factor\ interaction} \\
2 & 2 & \cdots & \cdots & 0 & 2 & 2 & \cdots & \cdots & 0 & \cdots\cdots & 2 & 2 & \cdots & \cdots & 0 & 2 \\
2 & 0 & \cdots & \cdots & 0 & 2 & 2 & \cdots & \cdots & 0 & \cdots\cdots & 2 & 2 & \cdots & \cdots & 2 & 2 \\
0 & 2 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & 0 & \cdots\cdots & 2 & 2 & \cdots & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 2 & \cdots\cdots & 2 & 2 & \cdots & \cdots & 2 & 2 \\
0 & 0 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & 2 & \cdots\cdots & 2 & 0 & \cdots & \cdots & 2 & 2 \\
0 & 0 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & 2 & \cdots\cdots & 0 & 2 & \cdots & \cdots & 2 & 2
\end{array}
}_{Quadratic\ Effect}
$$

$$
\left.
\underbrace{
\begin{array}{ccccc|ccccc|ccccc|c|ccccc|ccccc|ccccc}
\multicolumn{5}{c}{\overbrace{\hphantom{2\ 2\ \cdots\ \cdots\ 0}}^{2-factor\ interaction}} & \multicolumn{10}{c}{\overbrace{\hphantom{2\ 2\ \cdots\ \cdots\ 0\ 2\ \cdots\ \cdots\ \cdots\ 0}}^{3-factor\ interaction}} & & \multicolumn{15}{c}{\overbrace{\hphantom{2\ 1\ \cdots\ \cdots\ 1\ 2\ 2\ \cdots\ \cdots\ 1\ 2\ 2\ \cdots\ \cdots\ 1}}^{n-factor\ interaction}} \\
2 & 2 & \cdots & \cdots & 0 & 2 & 2 & \cdots & \cdots & 0 & 2 & \cdots & \cdots & \cdots & 0 & \cdots\cdots & 2 & 1 & \cdots & \cdots & 1 & 2 & 2 & \cdots & \cdots & 1 & 2 & 2 & \cdots & \cdots & 1 \\
1 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 0 & 2 & \cdots & \cdots & \cdots & 0 & \cdots\cdots & 1 & 2 & \cdots & \cdots & 1 & 2 & 1 & \cdots & \cdots & 1 & 2 & 2 & \cdots & \cdots & 2 \\
0 & 2 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 2 & \cdots & \cdots & 1 & 2 & 2 & \cdots & \cdots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 1 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 1 & 2 & 2 & \cdots & \cdots & 2 \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 2 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 2 & 2 & 1 & \cdots & \cdots & 2 \\
0 & 0 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & 2 & 0 & \cdots & \cdots & \cdots & 2 & \cdots\cdots & 1 & 1 & \cdots & \cdots & 2 & 1 & 1 & \cdots & \cdots & 2 & 1 & 2 & \cdots & \cdots & 2
\end{array}
}_{Linear+Quadratic\ Effect}
\right]
$$

(2.1)

Since in N^*; there are “n” rows. Considering these n-rows as treatments, we have $v^* = n$.

In incidence matrix N^*, for linear effect, among ${}^{n}c_2$ columns, in each column, element one occurs ${}^{n-1}c_1$ times while zero occurs (n-2) times, among ${}^{n}c_3$ columns, element one occurs ${}^{n-1}c_2$ while zero occurs (n-3) times in each column and so on. Also due to merging of blocks, ${}^{n}c_1$ columns are obtained in which one occurs (n-1) times and two occurs once. For quadratic effect, among ${}^{n}c_2$ columns, in each column, element two occurs ${}^{n-1}c_1$ times while zero occurs (n-2) times, among ${}^{n}c_3$ columns, element one occurs ${}^{n-1}c_2$ while zero occurs (n-3) times in each column and so on. While ${}^{n}c_1$ columns having high-level main effects were deleted. For linear and quadratic effect, in two-factor interaction, among $2\times{}^{n}c_2$ columns, in each column, elements one and two occur ${}^{n-1}c_1$ times while zero occurs (n-1) (n-2) times; in three-factor interaction, among $6\times{}^{n}c_3$ columns, elements one and two occur $3\times{}^{n-1}c_2$ times while zero occurs (n-1) (n-2) (n-3) times in each column and so on.

Thus we get

$$b^* = \left[\left\{\overbrace{{}^nC_2 + {}^nC_3 + \cdots\cdots + {}^nC_n + {}^nC_1}^{Linear\ effect}\right\} + \left\{\overbrace{{}^nC_2 + {}^nC_3 + \cdots\cdots + {}^nC_n}^{Quadratic\ effect}\right\}\right.$$

$$\left. + \left\{\overbrace{2\times{}^nC_2 + 6\times{}^nC_3 + \cdots\cdots + (2^n-2)\times{}^nC_n}^{Linear+Quadratic\ effect}\right\}\right]$$

$$\Rightarrow b^* = \left\{2^n - 1\right\} + \left\{2^n - (n+1)\right\} + \left\{{}^nC_0 + 6\times{}^nC_3 + \cdots\cdots + (2^n-2)\times{}^nC_n\right\}$$

$$\Rightarrow b^* = \left\{2^n - 1\right\} + \left\{2^n - (n+1)\right\} + \left\{{}^nC_n\, 2^n + {}^nC_{n-1}\, 2^{n-1} + \cdots\cdots + {}^nC_0\, 2^0 - {}^nC_{n-1}\, 2^{n-(n-1)} - {}^nC_0\, 2^0\right\}$$
$$-2\left\{{}^nC_n + {}^nC_{n-1} + \cdots\cdots + {}^nC_0 - {}^nC_0 - {}^nC_1\right\}$$

$$\Rightarrow b^* = \left\{2^n - 1\right\} + \left\{2^n - (n+1)\right\} + \left\{3^n - 1 - 2n\right\} - 2\left\{2^n - (n+1)\right\}$$

$$\Rightarrow b^* = 3^n - (n+1)$$

Also in N^*; In linear effect, ${}^nc_2$ blocks have block size 2, ${}^nc_3$ blocks have block size 3, and so on and due to merging of blocks ${}^nc_1$ blocks have block size (n+1). In quadratic effect, ${}^nc_2$ block have block size 4, ${}^nc_3$ blocks have block size 6, and so on. In linear + quadratic effect, $2\times{}^nc_2$ blocks have block size 3, $6\times{}^nc_3$ blocks have block sizes 4, 5 respectively and $(2^n\text{-}2)$ blocks have block sizes (n+1), (n+2), ……(n+n). Thus

$$k^* = [2, 2, \ldots\ldots, 2\,; 3, 3, \ldots\ldots, 3\,; \ldots\ldots\ldots\,; n, n, \ldots\ldots, n\,; (n+1), (n+1), \ldots\ldots, (n+1)\,; (n+2), (n+2), \ldots\ldots, (n+2)\,; \ldots\ldots\ldots\,; (n+n)\,]$$

Since we have considered rows as treatments and columns as blocks. In N^* there are $v^* = n$ treatments and $b^* = [3^n - (n+1)]$ blocks. In each row, one occurs $(3^{n-1} + n - 2)$ times and two occurs 3^{n-1} times. Thus row sum becomes

$$r^* = \left\{{}^{n-1}C_1 + {}^{n-1}C_2 + \cdots\cdots + {}^{n-1}C_{n-1} + (n+1)\right\} + 2\left\{{}^{n-1}C_1 + {}^{n-1}C_2 + \cdots\cdots + {}^{n-1}C_{n-1}\right\}$$

$$+3\left\{{}^{n-1}C_1 + 3\times{}^{n-1}C_2 + \cdots\cdots + (2^{n-1}-1)\,{}^{n-1}C_{n-1}\right\}$$

$$\Rightarrow r^* = \left(2^{n-1} + n\right) + 2\left(2^{n-1} - 1\right) + 3\left\{(3^{n-1} - 1) - (2^{n-1} - 1)\right\}$$

$$\Rightarrow r^* = 3^n + (n-2)$$

Now calculation of variance and efficiency can be done as follows

Since a block design is variance-balanced iff

$$C = \psi^* \left[I_v - \frac{1}{v} J_v J_v' \right] \tag{2.2}$$

Where, ψ^* is the unique non-zero eight value of the C -matrix with multiplicity (v-1).

The C-matrix for the design having incidence matrix given in (2.2) can be written as

$$C = \theta \begin{bmatrix} n-1 & & & \\ & \ddots & -1 & \\ & & \ddots & \\ & -1 & & \ddots \\ & & & & n-1 \end{bmatrix} \tag{2.3}$$

Where,

$$\theta = \Bigg[\overbrace{\left\{ \frac{n+2}{n+1} + \frac{{}^{n-2}C_0}{n} + \frac{{}^{n-2}C_1}{n-1} + \frac{{}^{n-2}C_2}{n-2} + \cdots\cdots + \frac{{}^{n-2}C_m}{n-m} \right\}}^{Linear\ Effect} + \overbrace{\left\{ 2^2 \left(\frac{{}^{n-2}C_0}{2n} + \frac{{}^{n-2}C_1}{2n-2} + \frac{{}^{n-2}C_2}{2n-4} + \cdots\cdots + \frac{{}^{n-2}C_m}{2n-2m} \right) \right\}}^{Quadratic\ Effect}$$

$$+ \overbrace{\left\{ \left(\underbrace{\frac{(2)^2 ({}^{n-3}C_0 \times {}^{n-2}C_{n-3})}{n+1} + \frac{(2)^2 ({}^{n-2}C_1 + {}^{n-2}C_0) + {}^{n-2}C_{n-4}}{n+2} + \cdots\cdots + \frac{(2)^2 ({}^{n-2}C_{n-2} + {}^{n-2}C_{n-3})}{n+(n-1)}}_{n^{th}\ order\ interaction} \right) \right.}^{Linear+Quadratic\ Effect}$$

$$+ \overbrace{\left(\underbrace{\frac{(2)^2 ({}^{n-3}C_0 \times {}^{n-2}C_{n-3}) + ({}^{n-3}C_{n-4} \times {}^{n-2}C_{n-3})}{n} + \frac{(2)^2\ {}^{n-2}C_{n-3} \times ({}^{n-3}C_1 + {}^{n-3}C_0) + ({}^{n-3}C_{n-3} \times {}^{n-2}C_{n-3})}{n+1} + \cdots\cdots + \frac{(2)^2\ {}^{n-2}C_{n-3} \times ({}^{n-3}C_{n-3} + {}^{n-3}C_{n-4})}{n+(n-3)}}_{(n-1)^{th}\ order\ interaction} \right)}^{Linear+Quadratic\ Effect}$$

$$+ \cdots\cdots\cdots\cdots + \left. \overbrace{\left(\underbrace{\frac{(2)^2\ {}^{n-2}C_{(n-(n-2))-2} \times \left({}^{(n-(n-2))-2}C_0 + {}^{(n-(n-2))-2}C_{(n-(n-2))-1} \right)}{n-(n-3)}}_{\{n-(n-2)\}^{th}\ order\ interaction} \right)}^{Linear+Quadratic\ Effect} \right\} \Bigg] ;$$

$$\text{where } m = 1, 2, \ldots\ldots \qquad (2.4)$$

Comparing (2.2) and (2.3)

$$\psi^* = n \times \theta \tag{2.5}$$

Hence the incidence matrix defined in eq. (2.1) of design D^* gives equi-replicated variance balanced with unequal block sizes.

We know that M-matrix is defined as

$$M = I - R^{-1} C \tag{2.6}$$

After simplification we get,

$$M = \begin{bmatrix} 1-\frac{\theta(n-1)}{r^*} & & & \\ & \ddots & \frac{\theta}{r^*} & \\ & & \ddots & \\ & \frac{\theta}{r^*} & & \ddots & \\ & & & & \frac{\theta(n-1)}{r^*} \end{bmatrix} \quad (2.7)$$

Since MJ = J, where J is the unit vector of order (v×1).

Also, M-matrix is given as

$$M = \mu^* I_v + \left[(1-\mu^*)/\sum_i r_i^*\right] J_v \left(r^*\right) J_v' \quad (2.8)$$

Where, μ^* is the loss of information, I_v is the identify matrix of order (v×v), J_v is unit vector of order (v×1) and $\sum_i r_i^*$ is the total number of observations.

On simplification we get

$$M = \begin{bmatrix} 1-\frac{r^*}{\sum_i r_i^*} & & & & \\ & \ddots & & -\frac{r^*}{\sum_i r_i^*} & \\ & & \ddots & & \\ & -\frac{r^*}{\sum_i r_i^*} & & \ddots & \\ & & & & 1-\frac{r^*}{\sum_i r_i^{*2}} \end{bmatrix} + \begin{bmatrix} \frac{r^*}{\sum_i r_i^*} & \cdots & \cdots & \cdots & \frac{r^*}{\sum_i r_i^*} \\ \vdots & \frac{r^*}{\sum_i r_i^*} & & \frac{r^*}{\sum_i r_i^*} & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \frac{r^*}{\sum_i r_i^*} & & \frac{r^*}{\sum_i r_i^*} & \vdots \\ \frac{r^*}{\sum_i r_i^{*2}} & \cdots & \cdots & \cdots & \frac{r^*}{\sum_i r_i^{*2}} \end{bmatrix} \quad (2.9)$$

Comparing (2.7) and (2.9) we get,

$$\mu^* = \left[\sum_i r_i^* - \frac{\theta(n-1)\sum_i r_i^*}{r^*} - r^*\right] \times \frac{1}{\sum_i r_i^* - r^*} = 1 - \frac{\theta \sum_i r_i^*}{(r^*)^2} \quad (2.10)$$

Where, θ is defined in (2.4).

Thus the design is efficiency balanced with unequal block sizes.

Example 2.1 Let n=3, then in 3^3-factorial design; theorem 2.1 yields an incidence matrix N^* as given below

$$N^* = \left[\underbrace{\begin{matrix} \overbrace{\begin{matrix}1&1&0\\1&0&1\\0&1&1\end{matrix}}^{2-factor\ interaction} & \vdots & \overbrace{\begin{matrix}1\\1\\1\end{matrix}}^{3-factor\ interaction} & \vdots & \overbrace{\begin{matrix}2&1&1\\1&2&1\\1&1&2\end{matrix}}^{{}^3C_1\ blocks\ occurs\ due\ to\ merging\ of\ main\ effects\ with\ n-factor\ interaction}\end{matrix}}_{Linear\ Effect} \left| \underbrace{\begin{matrix} \overbrace{\begin{matrix}2&2&0\\2&0&2\\0&2&2\end{matrix}}^{2-factor\ interaction} & \overbrace{\begin{matrix}2\\2\\2\end{matrix}}^{3-factor\ interaction}\end{matrix}}_{Quadratic\ Effect} \right|\right.$$

$$\left.\underbrace{\begin{matrix} \overbrace{\begin{matrix}2&2&1&0&1&0\\1&0&2&2&0&1\\0&1&0&1&2&2\end{matrix}}^{2-factor\ interaction} & \vdots & \overbrace{\begin{matrix}2&1&1&\vdots&2&2&1\\1&2&1&\vdots&2&1&2\\1&1&2&\vdots&1&2&2\end{matrix}}^{3-factor\ interaction}\end{matrix}}_{Linear+Quadratic\ Effect}\right]$$

Here,

$$v^* = 3, \quad b^* = 23, \quad r^* = 28, \quad k^* = [2,....,2\ ;\ 3,....,3\ ;\ 4,....,4\ ;\ 5,....,5\ ;\ 6]$$

The C- matrix for the design having incidence matrix given above can be written as

$$C = \frac{119}{15}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{2.11}$$

Also, we know that

$$C = \psi^* \left[I_3 - \frac{1}{3} J_3\, J_3' \right] \tag{2.12}$$

Where ψ^* is the unique non-zero eight value of the C-matrix with multiplicity 2.

Comparing (2.11) and (2.12)

$$\psi^* = \frac{119}{5} \tag{2.13}$$

Thus the design is Variance balanced.

Now the M-matrix of the above design is given as

$$M = \begin{bmatrix} 182/420 & & & \\ & \ddots & \frac{119}{420} & \\ & \frac{119}{420} & \ddots & \\ & & & 182/420 \end{bmatrix} \tag{2.14}$$

Obviously, this matrix satisfies the condition of efficiency balanced design i.e. MJ=J; where J is the v×1 vector of ones.

The efficiency factor is calculated using the formula

$$M = \mu^* I_v + (1-\mu^*) J_v r^{*'} / \sum_i r_i^* \tag{2.15}$$

On simplification we get

$$M = \mu^* \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} + \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \tag{2.16}$$

Equating (2.14) and (2.16), we get

$$\mu^* = \frac{3}{20} \tag{2.17}$$

Thus the design is Efficiency balanced.

3 Optimality of the design

Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_{(v\text{-}1)}$ be non-zero eigenvalues of C_d matrix of design *d*. As we know that for variance balanced there will be only one non-zero eigenvalue with multiplicities (*v*–1) of C_d matrix of design d. That is, $\theta_1 = \theta_2 = \theta_3 = \ldots\ldots = \theta_{(v\text{-}1)} = \theta$ as C–matrix is positive semi-definite. Then,

A-Optimality: A design is A-Optimal if

$$\sum_{i=1}^{v-1} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{tr(C_d)}$$

D-Optimality: A design is D-Optimal if

$$\prod_{i=1}^{v-1} \frac{1}{\theta_i} \leq \prod_{i=1}^{v-1} \left\{ \sum_{i=1}^{v-1} \frac{1}{\theta_i} \Big/ (v-1) \right\}$$

E-Optimality: A design is E-Optimal if

$$\text{Min } (\theta_i) \leq \frac{tr(C_d)}{(v-1)}$$

Since the constructed variance balanced is A-optimal, D-optimal as well as E-optimal, hence the design is a “universally optimal”.

Example 3.1 Consider a variance balanced and efficiency balanced design obtained in the example 2.1 with parameters $v^* = 3$, $b^* = 23$, $r^* = 28$, $k^* = [2,\ldots, 2; 3,\ldots., 3; 4,\ldots., 4; 5,\ldots., 5; 6]$. The trace of C-matrix is comes out to be 238/5 and non-zero eight value of C-matrix is $\psi^* = \frac{119}{5}$ with multiplicity 2.

I) Checking A- Optimality

Here, the inequality

$$\sum_{i=1}^{3-1} \frac{1}{\psi_i^*} \geq \frac{(3-1)^2}{tr\,(C_d)} \quad \Rightarrow \quad \frac{10}{119} = \frac{10}{119}$$

holds true, which is the required condition of a variance balanced to be A-optimal, with equal replication and unequal block sizes. Thus the variance-balanced and efficiency balanced design constructed here is A-optimal.

II) Checking D-Optimality

Here, the inequality

$$\prod_{i=1}^{3-1} \frac{1}{\psi_i^*} \leq \prod_{i=1}^{3-1} \left\{ \sum_{i=1}^{3-1} \frac{1}{\psi_i^*} \Big/ (3-1) \right\} \quad \Rightarrow \quad \left(\frac{5}{119}\right)^2 = \left(\frac{5}{119}\right)^2$$

holds true, which is the required condition of a variance balanced to be D-optimal, with equal replication and unequal block sizes. Thus the variance-balanced and efficiency balanced design constructed here is D-optimal.

III) Checking E-optimality

Here, the inequality

$$\text{Min}\,(\psi_i^*) \leq \frac{tr\,(C_d)}{(3-1)} \quad \Rightarrow \quad \frac{119}{5} = \frac{119}{5}$$

holds true, which is the required condition of a variance balanced to be E-optimal, with equal replication and unequal block sizes. Thus the variance-balanced and efficiency balanced design constructed here is E-optimal.

Since the constructed variance balanced is A-optimal, D-optimal as well as E-optimal, hence the design is a “universally optimal”.

CONCLUSION

Since in many experimental situations, however, block designs with unequal block sizes and/or with unequal replications may be required. Here our main purpose is to study the universal optimality of block designs with unequal block sizes. In this research, we have significantly shown that the constructed variance-balanced and efficiency balanced designs are universally optimum. It is also expected that the results of our research in this area, appear to have the potential of being useful to the researchers engaged in this particular area of interest.

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